APPLICATIONS OF THE PROJECTIVE-ITERATIVE VERSIONS OF FINITE ELEMENT METHOD IN PROBLEMS OF DAMAGE FOR ENGINEERING STRUCTURES

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ABSRACT
Efficient projective-iterative versions of the finite element method (FEM) are developed to study structures with defects in the form of various breakdowns of continuity and inclusions under elastoplastic deformation. Corresponding numerical schemes are constructed. Used versions of FEM offer a considerable (several-fold) reduction in the time of computation of stress and strain fields in such structures. Schemes of successive approximations that account for plastic deformation are constructed. Developed calculated schemes may be used for simulation of the behavior various actual engineering structures and for diagnostics and monitoring in different problems of maintenance for structures with defects.

Keywords: finite element method (FEM), damage, holes, stress, strain, elastoplastic problem, carrying capacity, maintenance, diagnostics, monitoring.

1. INTRODUCTION

Real structures of modern techniques have defects – damages of various types: breakdowns of continuity – discontinuities (holes), initial geometrical imperfections et al.
They have a dominant role for problems of maintenance of structures. The causes of damages are different. They appear during both manufacture and service of structures. Occurrence of damage is necessarily preceded by loss of homogeneity of stress and strained state.
The behavior of such systems should be studied with allowance for material nonlinearity (plasticity, creep). Plastic deformation adds peculiar effects. Various structures of modern technique are plate-shell systems.
The numerical variational-grid finite element method (FEM) is an efficient numerical method for solution of various problems of deformation and critical states of plate-shell structures. The projective-iterative schemes of numerical FEM implementation considered in this article offer a far (several times) shorter running time, which is of importance in numerical simulation of complex process of nonlinear deformations of nonhomogeneous structures.

In this report (article) projective-iterative schemes of FEM implementation are the elastoplastic deformation of plate structural elements with different holes is studies. Plastic zone are determined and hole interaction is investigated. These investigations may be used for solution of maintenance problems of various plate-shell structures.

2. THEORETICAL FOUNDATION OF PROJECTIVE-ITERATIVE VERSION OF FINITE ELEMENT METHOD

Let a bounded-below functional *F(u)*, $\inf_{u \in \Omega} F(u) = F^* > -\infty$, be defined on some set Ω of Hilbert space. Let it be desired to minimize F(u) on Ω:

$$F(u) \to \inf, \; u \in \Omega. \qquad (1)$$

Consider the variant of the projective-iterative method of solution of the problem (1) which is based on the application of the pointwise relaxation method to the solution of a sequence of conditional minimization problems [2-5].

The basic idea of the projective-iterative FEM version is that the original extremum problem (1) is approximated, using the FEM, by a sequence of discrete extremum problems (n = 1, 2,…) for multivariable functions. Each of the problems, starting with some number $n = N$, is solved by the method of successive overrelaxation using the special procedure. For problems where stress concentration occurs, it is advisable to use adaptive grids. For example we consider plate with rectangular hole.

To construct an adaptive grid from rectangular finite elements, the following algorithm is used:

- in the region ω under consideration, a square grid of spacing $h_n > 0$ ($n = N$) formed by the straight lines $x_i = ih_n$, $y_i = jh_n$, $0 < i < N_x$, $0 < j < N_y$, $N_x = 2a/h_n$, $N_y = 2b/h_n$ is introduced (Fig. 1a);
- a zone of assumed stress concentration of length l_1 and width l_2 is specified (Fig. 1a) and an integer $d > 0$ is specified, and the zone of assumed stress concentration is partitioned by straight lines parallel to the *Ox* and *Oy* axes spaced by distance $h_n^d = h_n/d$ (Fig. 1b).

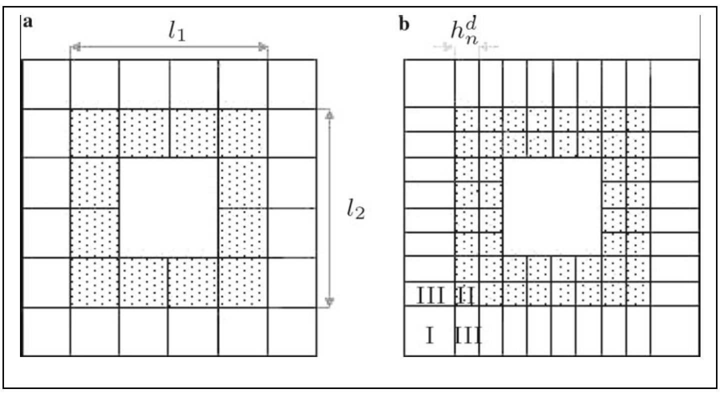

Figure 1. Construction of adaptive grid

As a result of these operations, we obtain a finite-element adaptive grid of dimension

$$\left(N_x + 1 + (d-1)C_x\right) \times \left(N_y + 1 + (d-1)C_y\right), \qquad (2)$$

where C_x and C_y are the numbers of elements in the stress-concentration zone along the *Ox* and *Oy* axes, respectively; a sequence of nested adaptive grids is obtained by triangulating the finite elements along the Ox and Oy axes with step $h_{n+1} = h_n / 2$ outside the stress concentration zone and with step $h_{n+1}^d = h_n^d / 2$ therein.

Thus, on each of finite-element grid we will obtain a set of finite elements of different dimensions: a square finite element of type I, a square finite element of type II whose sides are d times smaller than those of the element of type I, and a rectangular finite element of type III whose side ratio is equal to d (Fig. 1b).

The problem algorithm based on the application of the projective-iterative FEM version on the sequence of adaptive grids is as follows:

- for given geometrical parameters of the problem and a given computational accuracy ε, relaxation factor ω, and initial grid spacing h_n ($n = N$), an initial adaptive grid of the dimension (9) is constructed;
- an initial approximation is specified for $n = N$;
- the functional of the problem (1) is approximated by a multivariable function, which is minimized using the method of successive overrelaxation;
- the number k_n of the approximations to be constructed in the n-th step of the projective-iterative process is chosen equal to the least integer k that satisfies the inequality $\left\| z_n^{(k)} - z_n^{(k-1)} \right\| \le \varepsilon$, where $z_n^{(k)}$ is the approximate solution of the n-th finite-dimensional problem;
- the solution obtained on the n-th adaptive grid is interpolated to a finer (n+1)-th finite-element adaptive grid and used as an initial approximation to the point of minimum for the corresponding multivariable function.

It is of interest to study the effect of the shape of a finite element on the computational efficiency of the projective-iterative FEM-implementation scheme. We consider linear and bilinear approximations in the projective-iterative process (accordingly triangular and rectangular finite elements) in the problem of determining the stress and strain fields of a elastic plate with a rectangular or circular hole [5]. Projective-iterative FEM schemes for rectangular and triangular finite elements are developed, and the efficiencies of the corresponding computational algorithms are compared. The use of rectangular finite elements in the projective-iterative process reduces the running time in comparison with the use of triangular finite elements, and the stresses obtained using the above types of finite elements differ from one another by no more than 1,7%.

3. SOLUTION METHODS FOR ANALYSIS OF THE ELASTOPLASTIC PROBLEMS

Let us give a brief characteristic of solution methods for elestoplastic problems [5,6].

In the method of variable elastic parameters (MVEP), an iterative process is constructed, and in each approximation an elasticity problem with variable modulus of elasticity $\bar{E}$, shear modulus $\bar{G}$, and Poisson's ratio $\bar{\nu}$ is solved. In deformation theory, the following relations hold

$$\varepsilon_{ij} = \frac{1}{\bar{E}}\left(\left(1+\bar{\nu}\right)\sigma_{ij} - 3\bar{\nu}\sigma\delta_{ij} \right),$$

$$\bar{E} = 2\bar{G}\left(1+\bar{\nu}\right), \quad \bar{G} = \frac{1}{2\psi}, \quad \bar{\nu} = \frac{E\psi - 1 + 2\nu}{2E\psi + 1 - 2\nu}, \quad \psi = \frac{3\varepsilon_i}{2\sigma_i}, \qquad (3)$$

where σ_{ij} and ε_{ij} are the stresses and strains, σ_i and ε_i are the stress and strain intensities.

In a first approximation, we set $\bar{E} = E$, $\bar{\nu}_1 = \nu$ and determine σ_{ij1}, σ_{i1}, ε_{i1}, ψ_1. In a second approximation, we determine $\bar{E}_2 = \sigma_{i1}/\varepsilon_{i1}$, $\bar{\nu}_2$ (3) at ψ_1, and so on. The process is run until two successive approximations coincide to within a given accuracy, and the points determining σ_{im} and ε_{im} should be close to the deformation curve, $\bar{E}_m \approx \bar{E}_{m-1}$. The scheme of successive approximations with the use of the material deformation curve is shown in Fig. 2 (σ_s – yield strength).

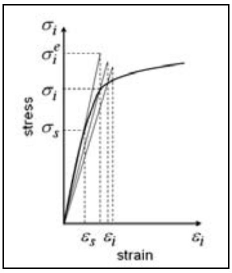

Figure 2. Scheme of successive approximations for MVEP

In the method of additional loads, additional quantities F_i^o and p_i^o are added to the external loads F_i and p_i in the equilibrium equations and boundary conditions:

$$\sigma_{ij,j}^{e} + F_i + F_i^o = 0 \,; \ \sigma_{ij} n_j = p_i + p_i^o \,,$$

$$F_i^o = -\left[\left(1 - \frac{1}{2G\psi}\right) s_{ij}^{e}\right]_{,j} \,; \quad p_i^o = \left(1 - \frac{1}{2G\psi}\right) s_{ij}^{e} n_j \,, \qquad (4)$$

where s_{ij} are stress deviators, and the superscript "e" denotes the quantities for an elastic body whose deformations are the same as in the elastoplastic body.

In a first approximation, we set $F_i^o = p_i^o = 0$ and solve the elasticity problem to determine σ_{ij1}^{e} and σ_{i1}^{e} (on the deformation curve, σ_{i1} and ε_{i1} correspond to σ_{i1}^{e}). Then the first-approximation parameter ψ_1 and stresses σ_{ij1}^{e} are determined as

$$\psi = \frac{3\varepsilon_i}{2\sigma_i} = \frac{(1+\nu)\sigma_{i1}^{e}}{E\sigma_{i1}} \,, \ \sigma_{ij1} = \frac{\sigma_{ij1}^{e}}{2G\psi_1} + \sigma^{e}\left(1 - \frac{1}{2G\psi_1}\right)\delta_{ij} \,. \quad (5)$$

In a second approximation, we solve the elasticity problem with the loads F_i^o and p_i^o (4), where at $\psi = \psi_1$, $s_{ij}^{e} = s_{ij1}$ and further calculations follow the scheme: $\sigma_{ij1}^{e} \rightarrow \sigma_{i2}^{e} \rightarrow \varepsilon_{i2}^{e} \rightarrow \sigma_{i2} \rightarrow \psi_2 \rightarrow F_{i2}^o, p_{i2}^o$. The process is run until two successive approximations coincide to within a given accuracy: $\sigma_{ijm} \approx \sigma_{ij(m-1)}$.

For the method of additional strains, the following relations hold:

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{o} \,, \ \varepsilon_i = \varepsilon_i^{e} + \varepsilon_i^{o} \,, \ \varepsilon_{ij}^{o} = \frac{E\psi - 1 - \nu}{E} s_{ij} \,, \quad (6)$$

where the superscript “e” means that ε_{ij} corresponds to a fictitious elastic body whose stresses σ_{ij} are the same as in the elastoplastic body.

In a first approximation, we set ψ = 1/(2G), $\varepsilon_{ij}^{o} = \varepsilon_{i}^{o} = 0$ and determine σ_{ij1}^{e}, ε_{ij1}^{e}, σ_{i1}, $\psi_1 = 3\varepsilon_{i1}/(2\sigma_{i1})$. In a second approximation, the elasticity problem with the strains ε_{ij}^{o} determined from (6) at ψ_1 is solved, and further calculations follow the scheme: $\varepsilon_{ij2}^{e} \to \varepsilon_{i2}^{e} \to \sigma_{i2} \to \psi_2 \to \varepsilon_{ij2}^{o}$. The process is run until two successive approximations coincide to within a given accuracy.

4. ANALYSIS OF ELASTOPLASTIC STRESS-AND-STRAIN STATE FOR PLATE STRUCTURAL ELEMENTS WITH DAMAGES IN THE FORM OF HOLES

Consider problems concerning the determination of the stress and strain fields in square plates structural element with different shaped holes with due regard given to the elastoplastic deformation of the material using both the projective-iterative FEM implementation schemes developed and the traditional FEM. Consider a plate with circular and rectangular (quadratic) holes. We will apply the variational methods [4,5] using MVEP. As a result, we will obtain an iterative process of successive approximations. In each approximation, a nonhomogeneous elasticity problem is solved, and the following functional is minimized

$$I(x,y) = \int_{\Omega} \frac{\bar{G}}{1-\bar{\nu}} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\bar{\nu}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{1-\bar{\nu}}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right] d\Omega - \int_{\Gamma} \left(p_x u + p_y v\right) dS, \quad (7)$$

where $\bar{G}$, $\bar{\nu}$ are the variable elastic parameters (3), u, v are the displacements, p_x, p_y are the displacement and loading vectors projections on the Ox and Oy axes, Ω – the region of plate, Γ – contour of boundary.

The strain intensity is

$$\varepsilon_i = \frac{2}{3}\left(\frac{1-\bar{\nu}+\bar{\nu}^2}{(1-\bar{\nu})^2}\left(\varepsilon_x + \varepsilon_y\right)^2 - 3\varepsilon_x\varepsilon_y + \frac{3}{4}\gamma_{xy}^2 \right)^{1/2}, \quad (8)$$

where ε_x, ε_y, γ_{xy} are strains.

The scheme of successive approximations for MVEP is shown in Fig. 2. Below are the results for a square plate composed of D16 aluminium alloy (σ_s = 200 MPa).

On fig. 3 the plastic zones in plate with circular and rectangular (quadratic) holes are shown. For fig. 3 a, b the load values q are 130 MPa (from the left) and 145 MPa (from the right). The radii of circular holes are 0,02 m and 0,03 m (for fig. 3 a, b respectively). The sides of quadratic holes are 0,06 m and 0,04 m (for fig. 3 a, b respectively). The distance between centers of holes is 0,12 m. The joining of plastic zones under load equal 145 MPa is realized. In this case the carrying capacity falls because of rigidity decrease. This characterizes the exhaustion of carrying capacity. It is possible the distributions of stresses σ_x and σ_y in various cross-sections of plates to construct [5]. These problems may be solved using the method of additional loads too.

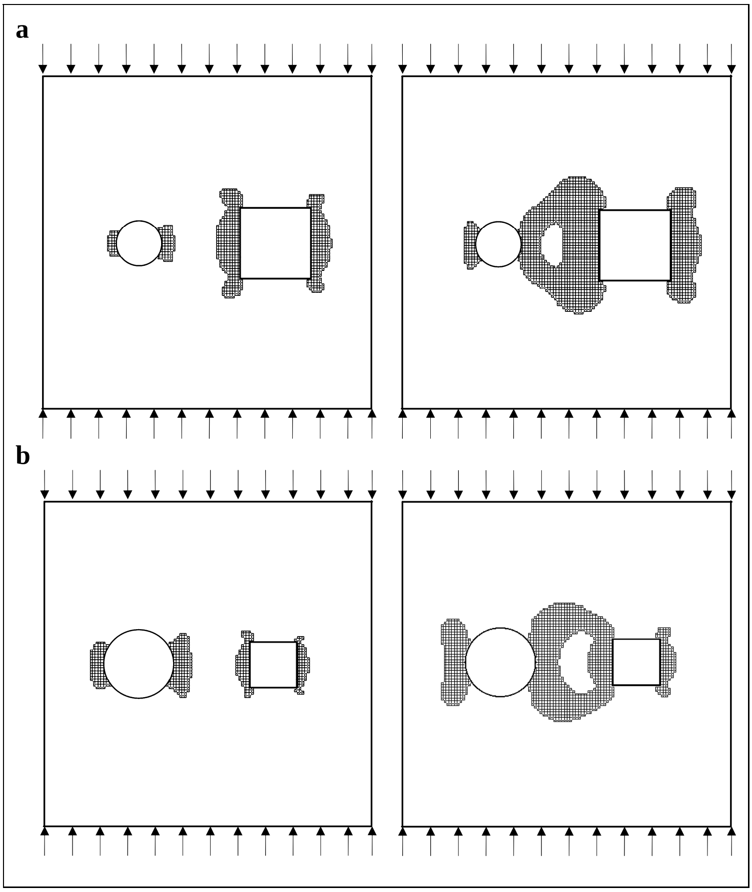

Figure 3. Zones of plastic strains for plate structural elements with two holes.

The calculation of the stress and strain fields for the elastoplastic deformation of the plate with these holes at the basis of the computational schemes developed offers a 56-fold reduction in running time in comparison with the traditional FEM.
The solution of the elastoplastic problem for one rectangular or circular hole using analytical or finite difference methods differs from the solution based on the FEM by no more than 3%.
The theoretical solution for the elasticity problem is in good agreement with the experimental data obtained using the photoelasticity method.
Schemes of calculation based on projective-iterative schemes of FEM realizations make possible consider several holes of various forms for materials with various deformation diagrams.

5. CONCLUSIONS

The projective-iterative schemes of numerical finite element method implementation have been proposed. The application of these schemes to the solution of the elastoplastic problems of deformation of nonhomogeneous rectangular or another form plate structural element with defects in the form of breakdowns of continuity: variously shaped (rectangular, circular, elliptic) holes demonstrates their high efficiency in terms of saving running-time in

comparison with the traditional FEM; in the various considered problems, their use offers a to 70-fold reduction in running time. This is of great importance in numerical simulation of the deformation and failure of plate-shell structural elements, where one has to solve a large number of similar problems using the FEM.
Elastoplastic deformation is accounted for by the construction of some processes of successive approximations. In each approximation, a nonhomogeneous elasticity problem is solved using projective-iterative FEM-implementation schemes. These schemes are efficient in the study of the elastoplastic deformation of plate-shell systems with holes and other nonhomogeneities such as various geometrical imperfections, inclusions, and reinforcements. The numerical simulation schemes developed and the computational algorithms implemented allow one to study defects (holes) interaction, construct plastic zones and solve the problems of carrying capacity.
Developed calculated schemes may be used for investigation of various behavior problems of different engineering structures composed from plates, shells and various thinwalled elements. These calculated schemes may be used for monitoring and diagnostics in different problems for maintenance of various structures with defects.

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